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Bundle functors and fibrations

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Abstract

We give an account of bundle-functors and star-bundle-functors (known from differential geometry) in terms of fibered categories.

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Introduction

The notions of bundle, and bundle functor, are useful and well exploited notions in topology and differential geometry, cf. e.g. [12], as well as in other branches of mathematics. The category theoretic set up relevant for these notions is that of fibred category, likewise a well exploited notion, but for certain considerations in the context of bundle functors, it can be carried further. In particular, we formalize and develop, in terms of fibred categories, some of the differential geometric constructions: tangent- and cotangent bundles, (being examples of bundle functors, respectively star-bundle functors, as in [12]), as well as jet bundles (where the formulation of the functorality properties, in terms of fibered categories, is possibly new).

Part of the development in the present note was expounded in [11], and is repeated almost verbatim in the Sections 2 and 4 below. These sections may have interest as a piece of pure category theory, not referring to differential geometry.

1 Basics on Cartesian arrows

We recall here some classical notions, cf. e.g. [2] or [17].

Let $\pi : \mathcal{X} \to \mathcal{B}$ be any functor. For $\alpha : A \to B$ in $\mathcal{B}$, and for objects $X, Y \in \mathcal{X}$ with $\pi(X) = A$ and $\pi(Y) = B$, let $\hom_\alpha(X, Y)$ be the set of arrows $h : X \to Y$ in $\mathcal{X}$ with $\pi(h) = \alpha$.

The *fibre* over $A \in \mathcal{B}$ is the category, denoted $\mathcal{X}_A$, whose objects are the $X \in \mathcal{X}$ with $\pi(X) = A$, and whose arrows are arrows in $\mathcal{X}$ which by π map to 1_A; such arrows are called *vertical* (over A). The hom functor of $\mathcal{X}_A$ is denoted $\hom_A$.

Let h be an arrow $X \to Y$, and denote $\pi(h)$ by $\alpha : A \to B$, where $A = \pi(X)$ and $B = \pi(Y)$. For any arrow $\xi : C \to A$, and any object $Z \in \mathcal{X}_C$, post-composition with $h : X \to Y$ defines a map

$$h_* : \hom_\xi(Z, X) \to \hom_{\xi.\alpha}(Z, Y).$$

(we compose from left to right). Recall that h is called *Cartesian* (with respect to π) if this map is a bijection, for all such ξ and Z. It is easy to see that Cartesian arrows compose, and that isomorphisms are Cartesian. In particular, the Cartesian arrows form a subcategory of $\mathcal{X}$. Also, a vertical Cartesian arrow is an isomorphism.

Example. The following is a fundamental example, which will also be the origin for some of the applications that we present. Let $\mathcal{B}$ be any category, and let $\mathcal{B}^2$ be the category of arrows in $\mathcal{B}$,

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so the arrows in $\mathcal{B}^2$ are the commutative squares in $\mathcal{B}$. Let $\partial_1 : \mathcal{B}^2 \to \mathcal{B}$ be the functor which to an arrow assigns its codomain. Then a commutative square in $\mathcal{B}$ is a Cartesian arrow in $\mathcal{B}^2$ (with respect to $\partial_1 : \mathcal{B}^2 \to \mathcal{B}$) precisely when the square is a pull-back.

The property of being a Cartesian arrow is clearly a kind of universal property. There is a weaker notion of when an arrow h as above is *pre-Cartesian*[1], namely that, for any $Z \in \mathcal{X}_A$, post-composition with h defines a bijection

$$h_* : \hom_A(Z, X) \to \hom_\alpha(Z, Y).$$

The property of being a pre-Cartesian arrow is clearly a universal property. In fact, to say that $h : X \to Y$ is preCartesian over α can be expressed by saying that h is terminal in a certain "relative comma-category" $\mathcal{X}_A \downarrow_\alpha Y$ whose objects are arrows in $\mathcal{X}$ over α with codomain Y, and whose arrows are arrows in $\mathcal{X}_A$ making an obvious triangle commute.

Remark. There are dual notions of coCartesian and pre-coCartesian arrows; they will not play much role in the following, except that we at one point shall consider the latter notion; thus, if $\alpha : A \to B$ in $\mathcal{X}$, and $X \in \mathcal{X}_A$, we have another "relative comma-category" $X \downarrow_\alpha \mathcal{X}_B$ whose objects are arrows in $\mathcal{X}$ over α with domain X, and whose arrows are arrows in $\mathcal{X}_B$ making an obvious triangle commute. Then a pre-coCartesian arrow over α with domain X is by definition an initial object in this category.

Clearly, if h is Cartesian, then it is also pre-Cartesian. Also Cartesian arrows over α, with given codomain Y, are unique up to unique vertical map (necessarily invertible) in $\mathcal{X}_A$; the same applies to pre-Cartesian arrows.

If h is Cartesian, the injectivity of h_* implies the cancellation property that h is "monic w.r. to π", meaning that for parallel arrows k, k' in $\mathcal{X}$ with codomain X, and with $\pi(k) = \pi(k')$, we have that $k.h = k'.h$ implies $k = k'$.

For later use, we recall a basic fact:

Lemma 1.1. If $k = k'.h$ is Cartesian and h is Cartesian, then k' is Cartesian.

For pull-back squares, this is a well known property for the functor $\partial_1 : \mathcal{B}^2 \to \mathcal{B}$, cf. the Example above. The proof of the general case is similar.

The functor $\pi : \mathcal{X} \to \mathcal{B}$ is called a *fibration* if there are enough Cartesian arrows, in the following sense: for every $\alpha : A \to B$ in $\mathcal{B}$ and every $Y \in \mathcal{X}_B$, there exists a Cartesian arrow over α with codomain Y. Such arrow is called *a Cartesian lift* of α with codomain Y. A choice, for each arrow $\alpha : A \to B$ in $\mathcal{B}$ and for each $Y \in \mathcal{X}_B$, of a Cartesian lift of α with codomain Y, is called a *cleavage* of the fibration π. The domain of the chosen Cartesian arrow over α with codomain Y is sometimes denoted $\alpha^*(Y)$. We use cleavages mainly as a notational tool, to facilitate reading, but generally, we avoid cleavages.

The functor $\partial_1 : \mathcal{B}^2 \to \mathcal{B}$ (cf. the Example above) is a fibration precisely when $\mathcal{B}$ is a category with pull-backs; then ∂_1 is called "the codomain fibration". A cleavage for it amounts to a choice of pull-back diagrams in $\mathcal{B}$.

[1]called *Cartesian* in the earlier literature (Grothendieck et.al); we follow the terminology mostly in use now, see e.g. [2] or [17].

2 The "factorization system" for a fibration

Let $\pi : \mathcal{X} \to \mathcal{B}$ be a fibration, and let $z : Z \to Y$ be an arrow in $\mathcal{X}$. Let $h : X \to Y$ be a Cartesian arrow over $\alpha := \pi(z)$. By the universal property of the Cartesian arrow h, there is a unique vertical $v : Z \to X$ with $z = v.h$.

Thus, every arrow z in $\mathcal{X}$ may be written as a composite of a vertical arrow followed by a Cartesian arrow (Cartesian arrows, we like to think of as being "horizontal"). And, crucially, this decomposition of z is unique modulo a *unique* vertical isomorphism. Or, equivalently, modulo a unique arrow which is at the same time vertical and cartesian. (Recall that for vertical arrows, being Cartesian is equivalent to being an isomorphism (= invertible).) This means that every arrow z in $\mathcal{X}$ may be represented by a pair (v, h) of arrows with v vertical and h cartesian, with $z = v.h$. We call such a pair a "vh composition pair", to make the analogy with vh spans, to be considered below, more explicit. Two such pairs (v, h) and (v', h') represent the same arrow in $\mathcal{X}$ iff there exists a vertical cartesian (necessarily unique, and necessarily invertible) s such that

$$v.s = v' \text{ and } s.h' = h. \tag{1.1}$$

We say that (v, h) and (v', h') are *equivalent* if this holds. The composition of arrows in $\mathcal{X}$ can be described in terms of representative vh composition pairs, as follows. If z_j is represented by (v_j, h_j) for $j = 1, 2$, then $z_1.z_2$ is represented by $(v_1.w, k.h_2)$, where k is cartesian over $\pi(h_1)$ and w is vertical, and the square displayed commutes:

$$\begin{tikzcd}
\cdot \arrow[d, "v_1"'] & & \\
\cdot \arrow[r, "h_1"] \arrow[d, "w"'] & \cdot \arrow[d, "v_2"] & \\
\cdot \arrow[r, "k"'] & \cdot \arrow[r, "h_2"'] & \cdot
\end{tikzcd} \tag{1.2}$$

Such k and w exists (uniquely, up to unique vertical cartesian arrows): construct first k as a Cartesian lift of $\pi(h_1)$ with same codomain as v_2, then use the universal property of Cartesian arrows to construct w.

The arrows z_1 and z_2 may be inserted, completing the diagram with two commutative triangles, since $z_j = v_j.h_j$. But if we refrain from doing so, we have a blueprint for a succinct and choice-free description of the fibrewise dual $\mathcal{X}^*$ of the fibration $\mathcal{X} \to \mathcal{B}$, to be described in Section 4.

Note that a vh factorization of an arrow in $\mathcal{X}$ is much reminiscent of the factorization for an E-M factorization system, as in [2] I.5.5, say, with the class of vertical arrows playing the role of E, and the class of Cartesian arrows playing the role of M; however, note that not every isomorphism in $\mathcal{X}$ is vertical.

3 Construction of functors out of a fibered category

Let $\pi : \mathcal{X} \to \mathcal{B}$ be a fibration, and consider a functor $F : \mathcal{X} \to \mathcal{Y}$. Let $\underline{\mathcal{X}}$ denote the category of vertical arrows in $\mathcal{X}$. Then by restriction, F gives a functor $\underline{F} : \underline{\mathcal{X}} \to \mathcal{Y}$. The restriction of F (or of $\underline{F}$) to the fibre $\mathcal{X}_A$ is denoted F_A. Similarly, let $\overline{\mathcal{X}}$ denote the category over $\mathcal{B}$ consisting of the Cartesian arrows of $\mathcal{X}$ only, and let $\overline{F}$ denote the restriction of F to $\overline{\mathcal{X}}$. Then F gives rise to the following data:

1) for each $A \in \mathcal{B}$, a functor $F_A : \mathcal{X}_A \to \mathcal{Y}$

2 a functor $\overline{F} : \overline{\mathcal{X}} \to \mathcal{Y}$.

Since the F_As and $\overline{F}$ are restrictions of the same functor F, it is clear that we have the properties

3) if s is vertical over A, and Cartesian, $F_A(s) = \overline{F}(s)$

4) Given a commutative square in $\mathcal{X}$, with v and w vertical and with h and k Cartesian (over $\alpha : A \to B$, say)

$$\begin{tikzcd} \cdot \arrow[r, "k"] \arrow[d, "v"'] & \cdot \arrow[d, "w"] \\ \cdot \arrow[r, "h"'] & \cdot \end{tikzcd}$$

Then $F_A(v).\overline{F}(h) = \overline{F}(k).F_B(w)$.

Theorem 3.1. Given functors $F_A : \mathcal{X}_A \to \mathcal{Y}$ (for all $A \in \mathcal{B}$), and given a functor $\overline{F} : \overline{\mathcal{X}} \to \mathcal{Y}$ as in 1) and 2), and assume that the conditions 3) and 4) hold. Then there exists a unique functor $F : \mathcal{X} \to \mathcal{Y}$ with restrictions F_A to the fibres $\mathcal{X}_A$ and with restriction $\overline{F}$ to the Cartesian arrows.

If $\mathcal{Y}$ comes with a functor to $\mathcal{B}$, and if for all $A \in \mathcal{B}$, F_A factors through $\mathcal{Y}_A \subseteq \mathcal{Y}$, then the constructed F is a functor over $\mathcal{B}$.

Proof. Given an arrow x over $\alpha : A \to B$, say. Since $\mathcal{X} \to \mathcal{B}$ is a fibration, x admits a vh factorization $x = v.h$ with v vertical over A and h Cartesian over α, so we are forced to define $F(x) := F_A(v).\overline{F}(h)$. To see that this F is well defined, we consider another possible vh factorization $x = v'.h'$. It compares with the given $v.h$ by a vertical Cartesian s with $v' = v.s$ and $h = s.h'$. We have

$$F_A(v).\overline{F}(h) = F_A(v).\overline{F}(s.h') = F_A(v).\overline{F}(s).\overline{F}(h'),$$

$$F_A(v').\overline{F}(h') = F_A(v.s).\overline{F}(h') = F_A(v).F_A(s).\overline{F}(h'),$$

using that $\overline{F}$ and F_A are functors. By condition 3), $\overline{F}(s) = F_A(s)$, so the two expressions agree. Let us next prove that the F constructed preserves composition of arrows, say $f_1.f_2$. Pick a vh factorization of f_1, say $f_1 = v_1.h_1$, and similarly for f_2. Interpolate a w and k as in (1.2); then use condition 4) for the interpolated square. –The last assertion is obvious. Q.E.D.

Even when, as in the last statement of the Proposition, the category $\mathcal{Y}$ is given as a category over $\mathcal{B}$, it is not assumed to be fibered over $\mathcal{B}$. But if $\mathcal{Y} \to \mathcal{B}$ happens to be a fibration, then given the family of functors $F_A : \mathcal{X}_A \to \mathcal{Y}_A$, the data of the functor $\overline{F}$ may be formulated in an alternative way, provided we assume given cleavages of both $\mathcal{X} \to \mathcal{B}$ and $\mathcal{Y} \to \mathcal{B}$. For then, to give the value of $\overline{F}$ on a Cartesian arrow $X' \to X$ over α, it suffices to give the value of $\overline{F}$ on the chosen Cartesian arrow $h : \alpha^*(X) \to X$ over α and with codomain X. This value is an arrow in $\mathcal{Y}$ over α

and with codomain $F_B(X)$, and as such factors uniquely by a vertical arrow $v_{\alpha,X}$ followed by the chosen Cartesian arrow over α with that codomain, thus

$$\begin{tikzcd} F_A(\alpha^*(X)) \arrow[d, "v_{\alpha,X}"'] \arrow[dr, "\overline{F}(h)"] & \\ \alpha^*(F_B(X)) \arrow[r] & F_B(X) \end{tikzcd}$$

with the bottom arrow the chosen Cartesian. (Note that $\overline{F}(h)$ need not be Cartesian; we did not assume that $\overline{F}$ preserves the property of being Cartesian.) So the data of $\overline{F}$ resides in the F_As, together with the vertical maps

$$v_{\alpha,X} : F_A(\alpha^*(X)) \to \alpha^*(F_B(X)). \tag{1.3}$$

(The $v_{\alpha,X}$ thus derived satisfy certain equations, in particular, for fixed α, $v_{\alpha,X}$ is natural in $X \in \mathcal{X}_B$; there are also equational conditions involving the comparison isomorphisms between $\alpha^* \circ \beta^*$ and $(\beta \circ \alpha)^*$. In terms of pseudofunctors sometimes used to present fibrations, v is a lax (or colax?) transformation between the pseudofunctors representing $\mathcal{X}$ and $\mathcal{Y}$, respectively. - We shall not enter into these conditions, since the conditions in the Theorem are clear enough.)

4 The dual fibration $\mathcal{X}^*$; comorphisms

The construction[2] presented in this Section is elementary. (In fact it is clear that it makes sense for categories and fibrations internal to an exact category.) It is is a direct generalization of the "star bundle" construction of [12] 41.1, where it is presented to account for the functorial properties of, say, the formation of cotangent bundles in differential geometry.

Given a fibration $\pi : \mathcal{X} \to \mathcal{B}$. We describe another category $\mathcal{X}^*$ over $\mathcal{B}$, the "fibrewise dual of $\mathcal{X} \to \mathcal{B}$", as follows: The objects of $\mathcal{X}^*$ are the same as those of $\mathcal{X}$; the arrows $X \to Y$ are represented by vh spans, in the following sense:

Definition 4.1. A *vh span* in $\mathcal{X}$ from X to Y is a diagram in $\mathcal{X}$ of the form

$$\begin{tikzcd} \cdot \arrow[r, "h"] \arrow[d, "v"'] & Y \\ X & \end{tikzcd} \tag{1.4}$$

with v vertical and h cartesian.

[2] The construction, in elementary terms, of the dual fibration can be distilled out of a more general construction [1] by Barwick et al. in the context of quasi-categories. I was unaware of their construction when I put a preliminary version [11] of the present paper on arXiv. I want to thank them for calling my attention to their work. The construction (for categories, not for quasi-categories) was apparently also known by Borceux, cf. Exercise 8.8.2 in [2] II.

The set of arrows in $\mathcal{X}^*$ from X to Y are equivalence classes of vh spans from X to Y, for the equivalence relation $\equiv$ given by $(v, h) \equiv (v', h')$ if there exists a vertical isomorphism s (necessarily unique) in $\mathcal{X}$ so that

$$s.v. = v' \text{ and } s.h = h'. \tag{1.5}$$

We denote the equivalence class of the vh span (v, h) by $\{(v, h)\}$. They are the arrows of $\mathcal{X}^*$; the direction of the arrow $\{(v, h)\}$ is determined by its Cartesian part h.

Composition has to be described in terms of representative pairs; it is in fact the standard composite of spans, but let us be explicit: If z_j is represented by (v_j, h_j) for $j = 1, 2$, then $z_1.z_2$ is represented by $(w.v_1, k.h_2)$, where k is Cartesian over $\pi(h_1)$ and w is vertical, and the square displayed commutes:

$$\begin{tikzcd}
\cdot \arrow[r, "k"] \arrow[d, "w"'] & \cdot \arrow[r, "h_2"] \arrow[d, "v_2"] & \cdot \\
\cdot \arrow[r, "h_1"'] \arrow[d, "v_1"'] & \cdot & \\
\cdot & &
\end{tikzcd} \tag{1.6}$$

Such k and w exists (uniquely, up to unique vertical cartesian arrows): construct first k as a cartesian lift of $\pi(h_1)$, then use the universal property of cartesian arrows to construct w. (The square displayed will then actually be a pull-back diagram, thus the composition described will be the standard composition of spans.)

Composition of vh spans does not give a definite vh span, but rather an equivalence class of vh spans. So referring to (1.6), the composite of $\{(v_1, h_1)\}$ with $\{(v_2, h_2)\}$ is defined by

$$\{(v_1, h_1)\}.\{(v_2, h_2)\} := \{(w.v_1, k.h_2)\}.$$

There is a functor π^* from $\mathcal{X}^*$ to $\mathcal{B}$; on objects, it agrees with $\pi : \mathcal{X} \to \mathcal{B}$; and $\pi^*(\{(v, h)\}) = \pi(h)$. Note that if $v : X' \to X$ is vertical, the vh span $(v, 1)$ represents a morphism $X \to X'$ in $\mathcal{X}^*$.

Clearly, a vertical arrow in $\mathcal{X}^*$ has a unique representative span of the form $(v, 1)$. So the fibres of $\pi^* : \mathcal{X}^* \to \mathcal{B}$ are canonically isomorphic to the duals of the fibres of $\pi : \mathcal{X} \to \mathcal{B}$, i.e. $(\mathcal{X}^*)_A \cong (\mathcal{X}_A)^{op}$; so $\mathcal{X}^*$ is "fibrewise dual" to $\mathcal{X}$ (but is not in general dual to $\mathcal{X}$, since the functor $\pi^* : \mathcal{X}^* \to \mathcal{B}$ is still a covariant functor). The arrows in $\mathcal{X}^*$, we call *comorphisms* in $\mathcal{X}$; it is ususally harmless to use the name "comorphism" also for a representing vh span (v, h).

There are two special classes of comorphisms: the first class consists of those comorphisms that can be represented by a pair $(v, 1)$ where 1 is the relevant identity arrow. They are precisely the vertical arrows for $\mathcal{X}^* \to \mathcal{B}$. – The second class consists of those comorphisms that can be represented by a pair $(1, h)$ where 1 is the relevant identity arrow. We shall see that these are precisely the cartesian morphisms in $\mathcal{X}^*$.

We first note that if (v, h) represents an arbitrary arrow in $\mathcal{X}^*$, then

$$(v, h) \in \{(v, 1)\}.\{(1, h)\}; \tag{1.7}$$

this is witnessed by the diagram

$$\begin{tikzcd}
\cdot \arrow[r, "1"] \arrow[d, "1"'] & \cdot \arrow[r, "h"] \arrow[d, "1"] & \cdot \\
\cdot \arrow[r, "1"'] \arrow[d, "v"'] & \cdot & \\
\cdot & &
\end{tikzcd}$$

since the upper left square is of the form considered in (1.6).

Proposition 4.2. An arrow in $\mathcal{X}^*$ is Cartesian iff it admits a vh representative of the form $(1, h)$. Any vh representative of such arrow is of the form (w, k) with w (vertical and) invertible.

Proof. In one direction, let $(1, h)$ represent a comorphism $Y \to Z$ over the arrow β in $\mathcal{B}$, and let (v, k) represent a comorphism $X \to Z$ over $\alpha.\beta$. We display these data as the full arrows in the following display (in $\mathcal{X}$ and $\mathcal{B}$):

$$\begin{tikzcd}
\cdot \arrow[d, "v"'] \arrow[dr, dotted, "k'"'] \arrow[drr, "k"] & & \\
X & Y \arrow[r, "h"'] & Z; \\
: & : & : \\
\cdot \arrow[r, "\alpha"'] & \cdot \arrow[r, "\beta"'] & \cdot
\end{tikzcd}$$

The dotted arrow k', with $k'.h = k$, comes about by using the universal property of the Cartesian arrow h in $\mathcal{X}$. Since k and h are Cartesian, then so is k', by Lemma 1.1. So (v, k') is a comorphism over α, and $(v, k').(1.h) \equiv (v, k)$, and using the cancellation property of Cartesian arrows, (v, k') is easily seen to represent the unique comorphism over $\alpha.\beta$ composing with $(1, h)$ to give (v, k). So $(1, h)$ is Cartesian in $\mathcal{X}^*$

In the other direction, let g be a cartesian arrow in $\mathcal{X}^*$. Let (w, k) be an arbitrary representative of g. Then by (1.7), $g = \{(w, 1)\}.\{(1, k)\}$. Since g is assumed cartesian in $\mathcal{X}^*$, and $\{(1, k)\}$ is cartesian by what is already proved, it follows from Lemma 1.1 that $\{(w, 1)\}$ is cartesian. Since it is also vertical, it follows that it is an isomorphism in $\mathcal{X}^*$, hence w is an isomorphism in $\mathcal{X}$. (And this proves the second assertion of the Proposition.) Since k is cartesian in $\mathcal{X}$, $w^{-1}.k$ is cartesian as well, and

$$(w, k) \equiv (1, w^{-1}.k),$$

so g has a representative of the claimed form. Q.E.D.

Proposition 4.3. The functor $\pi^* : \mathcal{X}^* \to \mathcal{B}$ is a fibration over $\mathcal{B}$

Proof. Let $\beta : A \to B$ be an arrow in $\mathcal{B}$, and let $Y \in \mathcal{X}_B$. Since $\mathcal{X} \to \mathcal{B}$ is a fibration, there exists in $\mathcal{X}$ a cartesian arrow h over β with codomain Y, and then the vh span $(1, h)$ represents, by the above, a cartesian arrow in $\mathcal{X}^*$ over β. Q.E.D.

The argument gives what may briefly be expressed: the Cartesian arrows of $\mathcal{X}$ are the same as the Cartesian arrows of $\mathcal{X}^*$.

Since $\mathcal{X}^* \to \mathcal{B}$ is a fibration, we may ask for its fibrewise dual $\mathcal{X}^{**}$:

Proposition 4.4. There is a canonical isomorphism over $\mathcal{B}$ between $\mathcal{X}$ and $\mathcal{X}^{**}$.

Proof. We describe an explicit functor $y : \mathcal{X} \to \mathcal{X}^{**}$. Let us denote arrows in $\mathcal{X}^*$ by dotted arrows; they may be presented by vh spans (v, h) in $\mathcal{X}$. We first describe y on vertical and cartesian arrows separately. For a vertical v in $\mathcal{X}$, say $v : X \to X'$, we have the vh span $(v, 1)$ in $\mathcal{X}$, which represents a vertical arrow $\overline{v} : X' \dashrightarrow X$ in $\mathcal{X}^*$; thus we have a vh span $(\overline{v}, 1)$ in $\mathcal{X}^*$, which in turn represents a vertical arrow $X \to X'$ in $\mathcal{X}^{**}$. This arrow, we take as $y(v) \in \mathcal{X}^{**}$. Briefly, $y(v) = ((v, 1), 1)$. – For a cartesian $h : X' \to Y$ (over β, say), we have a vh span $(1, h)$ in $\mathcal{X}$, which represents a horizontal arrow $\overline{h} : X' \dashrightarrow Y$ in $\mathcal{X}^*$ (cartesian over β); thus we have a vh span $(1, \overline{h})$ in $\mathcal{X}^*$, hence an arrow in $\mathcal{X}^{**}$, from X' to Y which we take as $y(h) \in \mathcal{X}^{**}$; briefly, $y(h) = (1, (1, h))$. The construction Theorem 3.1 can now be applied; thus for a general $f : X \to Y$ in $\mathcal{X}$, we factor it $v.h$ with v vertical and h cartesian, and put $y(f) := y(v).y(h)$. We leave to the reader to verify the conditions 3) and 4) of the Theorem, i.e. that a different choice of v and h gives an equivalent vh span in $\mathcal{X}^*$, thus the same arrow in $\mathcal{X}^{**}$.

Conversely, given an arrow $g : X \to Y$ in $\mathcal{X}^{**}$, represent it by a vh span in $\mathcal{X}^*$, $(\overline{v}, \overline{h})$,

$$\begin{tikzcd} X' \arrow[r, dotted, "\overline{h}"] \arrow[d, dotted, "\overline{v}"'] & Y \\ X & \end{tikzcd}$$

Since $\overline{v}$ is vertical, we may pick a representative of $\overline{v}$ in the form $(v, 1)$ with $v : X \to X'$, and since $\overline{h}$ is cartesian in $\mathcal{X}^*$, we may pick a representative of it if the form $(1, h)$, with $h : X' \to Y$ in $\mathcal{X}$. Then the composite $v.h : X \to Y$ makes sense in $\mathcal{X}$, and it goes by y to the given g. Q.E.D.

For simplicity of notation and reading, one sometimes assumes that one has a cleavage for a given fibration $\mathcal{X} \to \mathcal{B}$, i.e. a choice of Cartesian arrows; for α an arrow in $\mathcal{B}$ and Y an object in $\mathcal{X}$ over the codomain of α, the chosen Cartesian arrow over α is denoted $\alpha^*(Y) \to Y$. With such a cleavage, each equivalence class of vh composition pairs has a unique representative with one of these chosen arrows as h-part; and similarly for vh spans.

5 The codomain fibration, and bundle functors

Recall that if $\mathcal{B}$ is a category with pull-backs, then $\partial_1 : \mathcal{B}^2 \to \mathcal{B}$ is a fibration; the Cartesian arrows are the pull-back squares. This fibration is called the *codomain fibration*. Note that for $A \in \mathcal{B}$, the category $(\mathcal{B}^2)_A$ is the slice category $\mathcal{B}/A$.

For simplicity of notation, we assume in this Section a cleavage, which here amounts to a choice of pull-back diagrams, for any α and y with common codomain; then the following uses of the notation α^* is standard:

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "\alpha^*(y)"'] & \cdot \arrow[d, "y"] \\ A \arrow[r, "\alpha"'] & B \end{tikzcd} \quad \text{or with slight abuse:} \quad \begin{tikzcd} \alpha^*(Y) \arrow[r] \arrow[d] & Y \arrow[d, "y"] \\ A \arrow[r, "\alpha"'] & B. \end{tikzcd}$$

Note that we do not assume that $\alpha^*(Y) = y^*(A)$.

The identity functor $id_B : \mathcal{B} \to \mathcal{B}$ is likewise a fibration over $\mathcal{B}$ (this does not depend on $\mathcal{B}$ having pull-backs). When viewing $\mathcal{B}$ as being fibered over $\mathcal{B}$ in this way, it is sometimes useful to denote it $\mathcal{B}^1$, in analogy with $\mathcal{B}^2$; all arrows in $\mathcal{B}^1$ are Cartesian.

If $\mathcal{B}$ is the category of smooth manifolds, $\mathcal{B}^2$ is not a fibration; $\mathcal{B}$ does not have enough pull-backs. But one has a full subcategory $\mathcal{B}^{(2)}$ of $\mathcal{B}^2$ consisting of the submersions. It is a fibration over $\mathcal{B}$, again with pull-backs as Cartesian arrows.

An important class of functors over $\mathcal{B}$ are functors : $\mathcal{B}^1 \to \mathcal{B}^2$ (or, if $\mathcal{B}$ is the category of smooth manifolds, functors $\mathcal{B} \to \mathcal{B}^{(2)}$). We call such a functor a *bundle functor*. Thus, the data of such functor amounts to a functor $T_0 : \mathcal{B} \to \mathcal{B}$ plus a natural transformation $\pi : T_0 \to id_{\mathcal{B}}$ (whose instances are required to be submersions, in the case $\mathcal{B}^{(2)}$. Such data is called a bundle functor in [12], from where we have imported the terminology). Often, one does not notationally distinguish between T_0 and T, or one writes T for T_0 and π for the natural transformation. An example is the tangent bundle formation: If A is a smooth manifold, $T(A)$ is the tangent bundle of A, $\pi_A : T_0(A) \to A$ (we are disregarding for the moment the fibrewise vector space structure of the tangent bundle). Naturality of π says that for $\alpha : A \to B$

$$\begin{tikzcd} T_0(A) \arrow[r, "T_0(\alpha)"] \arrow[d, "\pi_A"'] & T_0(B) \arrow[d, "\pi_B"] \\ A \arrow[r, "\alpha"'] & B \end{tikzcd}$$

commutes. The bundle functor thus described is only Cartesian (i.e. preserves Cartesian arrows) when all squares of this form are pull-backs. (This square is clearly not a pull-back when T is the tangent bundle formation, unless α is a local diffeomorphism.)

6 The fibrewise dual of the codomain fibration

We describe $(\mathcal{B}^2)^*$, specializing the description in the Section 4. Explicitly, for this special case, its objects are likewise arrows in $\mathcal{B}$, and the arrows over $\alpha : A \to B$, from $x : X \to A$ to $y : Y \to B$, may be presented in the form of commutative diagrams η ("comorphisms" from x to y)

$$\begin{tikzcd}
\alpha^*(Y) \arrow[r] \arrow[d, "v"'] & Y \arrow[dd, "y"] \\
X \arrow[d, "x"'] \arrow[r, phantom, "\eta"] & \\
A \arrow[r, "\alpha"'] & B
\end{tikzcd} \tag{1.8}$$

where the rectangle is a pull-back; since we have chosen pull-backs, the presentation is unique if we insist that the top arrow is a chosen Cartesian, as suggested by the notation. Note that, given the x and y, as well as α, the information of the comorphism η resides in the map denoted v.

This kind of pull-back diagram was also considered in [18], under the name of "pull-back around α, x". By the general theory of Section 4, the comorphism $\eta : x \dashrightarrow y$ exhibited in (1.8) is Cartesian in $(\mathcal{B}^2)^*$ iff v is an isomorphism. This implies that it is a terminal object in the relative commacategory $(\mathcal{B}^2)^*_A \downarrow_\alpha y$. We may also ask the dual question: when is η pre-coCartesian, i.e. initial in the relative commacategory $x \downarrow_\alpha ((\mathcal{B}^2)^*)_B$? This is precisely to say that the diagram is a *distributivity* pull-back, in the sense of [18]; for, this means by definition that it is a terminal object in the category of "pull-backs around α, x" . The reason why our "initial" then is substituted for "terminal" in [18] is just that, in our set up, the A-fibre of $(\mathcal{B}^2)^*$ is *dual* to $\mathcal{B}/A$. When the functor $\alpha^* : \mathcal{B}/B \to \mathcal{B}/A$ has a right adjoint Π_α, then the y occurring in (1.8) is $\Pi_\alpha(x)$, and the v is the back adjunction $\alpha^*\Pi_\alpha(x) \to x$.

It is worthwhile to reformulate the description of the fibrations $\mathcal{B}^2 \to \mathcal{B}$ and $(\mathcal{B}^2)^* \to \mathcal{B}$ for the case where $\mathcal{B}$ is the category of sets, so that an object $\xi : X \to A$ in $\mathcal{B}^2$ or in $(\mathcal{B}^2)^*$ may be seen as a family $\{X_a \mid a \in A\}$ of sets: take $X_a := \xi^{-1}(a)$.

In this case, a morphism in $\mathcal{B}^2$ over $\alpha : A \to B$, from $X \to A$ to $Y \to B$, may be seen as a family of maps $\{f_a : X_a \to Y_{\alpha(a)} \mid a \in A\}$, and a morphism in $(\mathcal{B}^2)^*$ (i.e. a comorphism) over $\alpha : A \to B$ from $X \to A$ to $Y \to B$ may be seen as a family of maps $\{f_a : Y_{\alpha(a)} \to X_a \mid a \in A\}$. Let us write $f : X \dashrightarrow Y$ for such a comorphism, reserving the plain arrows for acual set maps. Composition of comorphisms is essentially just composition of maps: if $f : X \dashrightarrow Y$, as above, is a comorphism over $\alpha : A \to B$ and $g : Y \dashrightarrow Z$ similarly is a comorphism over $\beta : B \to C$, the composite of f followed by g is the comorphism $h : X \dashrightarrow Z$ over $A \to C$, given by $h_a(z) := f_a(g_{\alpha(a)}(z))$ for $z \in Z_{\beta(\alpha(a))}$.

Note that for $Y \to B$ and $\alpha : A \to B$, $\alpha^*(Y)$ is given by the A-indexed family $\{Y_{\alpha(a)} \mid a \in A\}$.

These set-theoretic descriptions do not depend on cleavages; on the contrary, suitably interpreted, reading $a \in A$ etc. as generalized elements (as in [8]), they describe the universal properties characterizing the objects or maps in question (even in more general categories). Similarly when reading objects in fibered categories as "generalized families" (as in [5] Chapter I).

7 Star bundle functors

As in two the previous sections, we consider a category $\mathcal{B}$ with pull-backs, so that we have two fibrations over $\mathcal{B}$, $\mathcal{B}^2$ and its fibrewise dual $(\mathcal{B}^2)^*$. We also have the trivial fibration $\mathcal{B}^1$ over $\mathcal{B}$. A *star bundle functor* (terminology from Kolář, Michor and Slovák, [12]) is now defined to be a functor S over $\mathcal{B}$ from $\mathcal{B}^1$ to $(\mathcal{B}^2)^*$. By the explicit description in the previous section, this amounts to the following data: for each $A \in \mathcal{B}$, an arrow $\pi_A : S_0(A) \to A$, and for each $\alpha : A \to B$, a pull-back diagram like (1.8),

$$\begin{tikzcd}
\alpha^* S_0(B) \arrow[r] \arrow[d, "v"'] & S_0(B) \arrow[dd, "\pi_B"] \\
S_0(A) \arrow[d, "\pi_A"'] & \\
A \arrow[r, "\alpha"'] & B.
\end{tikzcd}$$

More generally, if $\mathcal{X} \to \mathcal{B}$ is any fibration, a star-bundle functor with values in $\mathcal{X} \to \mathcal{B}$ is a functor over $\mathcal{B}$ from $\mathcal{B}^1$ to $\mathcal{X}^*$. The star bundle functors relevant for differential geometry considered in [12] have as $\mathcal{X}$ the full subcategory $\mathcal{B}^{(2)} \subseteq \mathcal{B}^2$ whose objects are the submersions between smooth manifolds.

The formation of cotangent bundles for manifolds is an example, to be described in the following Section. It is defined as the "fibrewise linear dual" of the tangent bundle, viewed as a vector bundle, i.e. as the composite of T with a "fiberwise duality" functor $\dagger$, whose categorical status will be described.

8 Vector bundles, and the cotangent bundle

The full generality of the present Section is probably that of fibrewise symmetric monoidal closed category, in the sense of [3] or [16], but we formulate things more concretely in terms of the fibered category $\mathcal{V} \to \mathcal{B}$ of vector bundles (over spaces in a suitable category $\mathcal{B}$ of, say, smooth manifolds). Thus $\mathcal{V}_A$ is the category of vector space objects in the category $\mathcal{B}/A$. This $\mathcal{V}$ comes with a forgetful functor over $\mathcal{B}$ from $\mathcal{V}$ to $\mathcal{B}^2$.

Again, we assume a cleavage, and the resulting notation like $\alpha^*(Y)$ for the chosen pull-back of a vector bundle Y along a smooth map α. We intend here to clarify the role of the notion of fibrewise linear dual of a vector bundle $X \to A$, which we denote $X^\dagger$ (refraining from using X^*, since the $*$ already has two meanings: α^* for pull-backs along α, and $\mathcal{Y}^*$ for the fibrewise dual fibration of a fibration $\mathcal{Y} \to \mathcal{B}$). Clearly, this dualization is a contravariant endofunctor on $\mathcal{V}_A$, for each $A \in \mathcal{B}$. (For vector *spaces*, the functor $\dagger$ is the standard contravariant dualization functor for vector spaces.)

Proposition 8.1. The fibrewise linear dualization functor $(-)^\dagger : \mathcal{V}_A \to (\mathcal{V}_A)^{op}$ extends canonically to a functor $\mathcal{V} \to \mathcal{V}^*$ over $\mathcal{B}$; it is a Cartesian functor.

Proof. Consider an arrow in $\mathcal{V}$ over α, meaning a commutative diagram

$$\begin{tikzcd} X \arrow[r, "t"] \arrow[d] & Y \arrow[d] \\ A \arrow[r, "\alpha"'] & B. \end{tikzcd}$$

with t fibrewise linear, so for each $a \in A$, the map t gives a linear map $t_a : X_a \to Y_{\alpha(a)}$, hence a linear $t_a^\dagger : (Y_{\alpha(a)})^\dagger \to X_a^\dagger$. But $(Y_{\alpha(a)})^\dagger = (Y^\dagger)_{\alpha(a)}$. Jointly, these $t_a^\dagger$ produce a map $\alpha^*(Y^\dagger) \to X^\dagger$ in $\mathcal{V}_A$, which is the vertical part of the desired comorphism; the horizontal part is the arrow $\alpha^*(Y^\dagger) \to Y^\dagger$ in the diagram defining $\alpha^*(Y^\dagger)$. – If the given square is a pull-back, each t_a is an isomorphism, hence so is $t_a^\dagger$, so in this case, the vertical part described is an isomorphism; therefore the comorphism described is Cartesian; this proves the last assertion. Q.E.D.

From this perspective, the cotangent bundle construction is a functor (over $\mathcal{B}$), namely the composite of the two functors

$$\mathcal{B} \xrightarrow{T} \mathcal{V} \xrightarrow{\dagger} \mathcal{V}^*;$$

both T and $\dagger$ are functors over $\mathcal{B}$, hence so is the composite. Here, $\mathcal{V}$ and $\mathcal{V}^*$ come with forgetful functors to $\mathcal{B}^2$ and $(\mathcal{B}^2)^*$, respectively. Composing with the forgetful functor $\mathcal{V}^* \to (\mathcal{B}^2)^*$ then gives a functor over $\mathcal{B}$, $\mathcal{B}^1 \to (\mathcal{B}^2)^*$, i.e. a star-bundle functor with values in $\mathcal{B}^2$.

The same argument as for the Proposition gives that the fibrewise linear dualization functor $(-)^\dagger : \mathcal{V}_A^{op} \to \mathcal{V}_A$ extends canonically to a functor $\mathcal{V}^* \to \mathcal{V}$ over $\mathcal{B}$; it is likewise Cartesian. (This does not depend on whether $V \to V^{\dagger\dagger}$ is an isomorphism.)

Remark 8.2. A more general description of a cotangent (star-bundle) functor exists in algebraic geometry, using Kähler differentials; the linear dual of it then may be used as a more generally applicable notion of tangent bundle. We give an account of this description, in terms of jet-bundles, in Section 12 below.

9 Strength

Let $\mathcal{B}$ be a category with finite limits and let $\mathcal{X} \to \mathcal{B}$ and $\mathcal{Y} \to \mathcal{B}$ be fibrations. We consider a functor $F : \mathcal{X} \to \mathcal{Y}$ over $\mathcal{B}$ (F is not assumed Cartesian). Then we shall consider a certain kind of structure on such a functor, which we call *fibrational strength*, or just *strength*.

For this, we introduce some notation. If $Q \in \mathcal{B}$ and $X \in \mathcal{X}_M$, we have an object $p^*(X) \in \mathcal{X}_{Q\times M}$, where $p : Q \times M \to M$ denotes the projection. This object in $\mathcal{X}_{Q\times M}$ we denote $Q \otimes X$. It comes equipped with a (Cartesian) morphism $Q \otimes X \to X$ over p.

Example. Let $\mathcal{B}$ be a category with finite limits. Then the codomain fibration $\partial_1 : \mathcal{B}^2 \to \mathcal{B}$ is a fibration, and pull-back squares in $\mathcal{B}$ are the Cartesian arrows in $\mathcal{B}^2$. If $\xi \in \mathcal{B}^2$, say $\xi : X \to M$, and $Q \in \mathcal{B}$, it is clear that $Q \otimes \xi$, as a map in $\mathcal{B}$, is just $Q \times \xi : Q \times X \to Q \times M$.

Let $F : \mathcal{X} \to \mathcal{Y}$ be a functor over $\mathcal{B}$, a *strength* on F consists in the following data: for $Q \in \mathcal{B}$ and $X \in \mathcal{X}_M$, one gives a morphism in $\mathcal{Y}_{Q\times M}$

$$t_{Q,M} : Q \otimes F(X) \to F(Q \otimes X),$$

natural in Q and X, and satisfying a unit- and associativity constraint w.r.to Q. (Example: for T the tangent bundle functor $\mathcal{B} \to \mathcal{B}^2$, there is a canonical strength: $t_{Q,X} : Q \otimes T(X) \to T(Q \times X)$ is the inclusion of the subbundle of *vertical* tangent vectors to $Q \times X$ (vertical w.r.to the projection $Q \times X \to Q$).)

Since F is a functor over $\mathcal{B}$, there is a canonical morphism $F(p^*(X)) \to p^*(F(X))$, i.e. $F(Q \otimes X) \to Q \otimes F(X)$ in $\mathcal{Y}_M$. It is invertible if f is Cartesian; and for F Cartesian, this inverse will be a strength structure $t_{Q,X}$ on F.

If $F : \mathcal{X} \to \mathcal{Y}$ and $G : \mathcal{Y} \to \mathcal{Z}$ are functors over $\mathcal{B}$, equipped with strengths t and s, respectively, one constructs out of t and s in an evident way a strength on the composite functor $G \circ F : \mathcal{X} \to \mathcal{Z}$. We obtain a 2-category: objects are categories fibered over $\mathcal{B}$, arrows are functors over $\mathcal{B}$ equipped with strength, and 2-cells are the vertical natural transformations between parallel functors over $\mathcal{B}$, compatible with the given strengths.

If $\mathcal{B}$ is the category of smooth manifolds, a map $h : Q \times M \to N$ in $\mathcal{B}$ may be seen as a smoothly parametrized family of smooth maps $h(q, -) : M \to N$ (with Q as the space of parameters), and a map $H : Q \otimes X \to X'$ over h may be seen as a Q-parametrized family H of maps in $\mathcal{X}$ from X to X', with the qth member of this family living over $h(q, -) : M \to N$.

A strength t of $F : \mathcal{X} \to \mathcal{Y}$ gives rise to a process transforming a parametrized family of maps in $\mathcal{X}$ to a similarly parametrized family of maps in $\mathcal{Y}$, as follows. Given a map $h : Q \times M \to N$ in $\mathcal{B}$, and given a map H in $\mathcal{X}$ over h, say $H : Q \otimes X \to X'$, then the composite

$$Q \otimes F(X) \xrightarrow{t_{Q,X}} F(Q \otimes X) \xrightarrow{F(H)} F(X')$$

is a map in $\mathcal{Y}$ over h; so F has transformed the Q-parametrized family H of maps from X to X' into a Q-parametrized family of maps $F(X) \to F(X')$. This property of F is called *regularity* in [12] 18.10.

The identity functor $\mathcal{B} \to \mathcal{B}$ is a fibration, denoted $\mathcal{B}^1$; for this fibration, $Q \otimes X = Q \times X$. Recall that a bundle functor is a functor $F : \mathcal{B}^1 \to \mathcal{B}^2$ over $\mathcal{B}$, thus for $X \in \mathcal{B}$, the object $F(X)$ in $\mathcal{B}^2$ is an arrow of the form $F(X) : F_0(X) \to X$ in $\mathcal{B}$, where F_0 is the composite of F with the domain formation $\partial_0 : \mathcal{B}^2 \to \mathcal{B}$. A fibrational strength t of such functor gives, for Q and X in M, an arrow $t_{Q,X}$ in $\mathcal{B}^2$ from $Q \times F(X)$ to $F(Q \times X)$, which amounts to a commutative square in $\mathcal{B}$ (really just a triangle) of the form

$$\begin{tikzcd} Q \times F_0(X) \arrow[r, "t''_{Q,M}"] \arrow[d, swap, "Q \times F(X)"] \arrow[dr, phantom, "t_{Q,X}"] & F_0(Q \times X) \arrow[d, "{F(Q \times X),}"] \\ Q \times X \arrow[r, swap, "id"] & Q \times X \end{tikzcd} \tag{1.9}$$

and the top map in this square (as Q and X range over $\mathcal{B}$) equips the endofunctor $F_0 : \mathcal{B} \to \mathcal{B}$ with a *tensorial* strength t'' in the sense of [7]. Vice versa, if such t'' make the squares like the above commute, these squares will constitute a fibrational strength t on F. (To say that the squares commute is in turn equivalent to saying that F, viewed as a natural transformation from F_0 to the identity functor on $\mathcal{B}$, is a strong natural transformation, in the sense of tensorial strength.)

Let $F : \mathcal{B}^1 \to \mathcal{B}^2$ be a bundle functor preserving finite products. Thus $F(Q \times B) \cong F(Q) \times F(B)$ by the canonical map. In particular (since $\partial_0 : \mathcal{B}^2 \to \mathcal{B}$ preserves products), $F_0(Q \times B) \cong F_0(Q) \times F_0(B)$. An example is where F is the tangent bundle functor (ignoring the fibrewise linear structure).

A particular bundle functor on $\mathcal{B}$ is the diagonal Δ associating to $B \in B$ the identity map $B \to B$. It terminal among bundle functors $\mathcal{B}^1 \to \mathcal{B}^2$.

A *section* of a bundle functor F is a natural transformation z (over $\mathcal{B}$) from $\Delta : \mathcal{B}^1 \to \mathcal{B}^2$ to F, thus to each $B \in \mathcal{B}$, $z_B : B \to F_0(B)$ is a section of $F(B) : F_0(B) \to B$. The zero section of a tangent bundle is an example.

Proposition 9.1. Let F be a finite-product preserving bundle functor equipped with a zero section. Then F carries a canonical (fibrational) strength.

Proof/Construction. By the above (cf. (1.9), it suffices to construct in $\mathcal{B}$ a map $t''_{Q,B} : Q \times F_0(B) \to F_0(Q \times B)$. This is taken to be the composite of $z_Q \times F_0(B) : Q \times F_0(B) \to F_0(Q) \times F_0(B)$ with the isomorphism $F_0(Q) \times F_0(B) \cong F_0(Q \times B)$. Q.E.D.

The notion of (fibrational) strength of a functor $F : \mathcal{X} \to \mathcal{Y}$ over $\mathcal{B}$, in the sense described here, generalizes the notion of "regularity" of a bundle functor, [12] 14.21 (and 18.10). The reason we change terminology from "regularity" to "strength" is to emphasize 1) that, in the abstract setting, it is a *structure* on the functor in question, not just a property, and 2) to tie it up with the notion of (monoidal, or tensorial) strength considered in the context of endofunctors F on a monoidal category $\mathcal{B}$, as in [7], (or [14], or [10] Section 2 for a recent account). Such a structure in turn is equivalent to a $\mathcal{B}$-enrichment of F, in case $\mathcal{B}$ is monoidal closed; cf. [7].

If $\mathcal{X} \to \mathcal{B}$ is the fibration, where $\mathcal{X}_B$ is the category of vector space objects in $\mathcal{B}/B$ (or group objects, or any other algebraic kind of structure), then there is a faithful forgetful functor $\mathcal{X} \to \mathcal{B}^2$ over $\mathcal{B}$, which is Cartesian, in particular, it preserves the formation $Q \otimes X$. So if the bundle functor F considered above factors through $\mathcal{X} \to \mathcal{B}^2$ (a "vector bundle functor"), then the $t''_{Q,B}$ constructed above is the underlying arrow of an arrow in $\mathcal{X}$, i.e. is fibrewise linear, and equippes the vector bundle functor F with a fibrational strength.

This in particular applies to the tangent bundle formation. The cotangent bundle functor likewise carries a canonical strength, by the following

Proposition 9.2. Let $\mathcal{X} \to \mathcal{B}$ be the category of vector bundles. Then if there is given a strength on $F : \mathcal{B} \to \mathcal{X}$, then there is canonically associated a strength on the star bundle functor $F^\dagger : \mathcal{B} \to \mathcal{X}^*$.

Proof. This follows since the dualization functor $\dagger : \mathcal{X} \to \mathcal{X}^*$ is Cartesian, and hence carries a canonical strength; and a composite of two functors with a strength has a strength. Note that the instantiations $t_{Q,B}$ of the strength described here are (vertical) maps $Q \otimes F^\dagger(B) \to F^\dagger(Q \times B)$ in $\mathcal{X}^*$, and hence as vector bundle maps (maps in $\mathcal{X}$) are maps $F^\dagger(Q \times B) \to Q \otimes F^\dagger B$; for, the fibre of $\mathcal{X}^*_{Q\times B}$ is $(\mathcal{X}_{Q\times B})^{op}$. Q.E.D.

10 Flow natural transformation

Consider a category $\mathcal{B}$ with Cartesian products, and consider an endofunctor $F : \mathcal{B} \to \mathcal{B}$ with a tensorial strenght $t''_{Q,B} : Q \times F(B) \to F(Q \times B)$ (for all Q and B in $\mathcal{B}$). If now $D \in \mathcal{B}$ is an exponentiable object, so $(D \times -) \dashv (D \pitchfork -)$, one derives, for all $B \in \mathcal{B}$, a map

$$\lambda_{D,B} : F(D \pitchfork B) \to D \pitchfork F(B),$$

namely the exponential transpose of the composite

$$D \times F(D \pitchfork B) \xrightarrow{t''_{D,D\pitchfork B}} F(D \times (D \pitchfork B)) \xrightarrow{F(ev)} F(B).$$

If $D_1 \to D_2$ is a map between exponentiable objects, one gets a map $D_2 \pitchfork B \to D_1 \pitchfork B$ (" $\pitchfork$ is contravariant in the first variable "), and then λ will be natural in the D_i's, in an evident sense. Also, $\lambda_{1,B} : F(1 \pitchfork B) \to 1 \pitchfork F(B)$ may be identified with the identity map on $F(B)$. So if the exponentiable object D is equipped with a point $0 : 1 \to D$, one obtains a commutative triangle

$$\pi \circ \lambda_{D,B} = F(\pi), \tag{1.10}$$

where π denotes $0 \pitchfork B$ or $0 \pitchfork F(B)$. So $\lambda_{D,B}$ is a map of bundles over $F(B)$.

(If $\mathcal{B}$ is Cartesian closed (or even just symmetric monoidal closed), all objects D are exponentiable, and therefore one has such a $\lambda_{D,X}$ for all D, X, and this data encodes strength of F in what may be called the "cotensorial" form of strength, cf. [6] or [10].)

For the following, we shall assume that $\mathcal{B}$ is a model for synthetic differential geometry, in particular, it may contain the category of smooth manifolds as a full subcategory, but it also contains some "infinitesimal objects", in particular, it contains an object D with the property that for any manifold M, $T(M) = D \pitchfork M$, and the base map $T(M) \to M$ is "evaluation at $0 \in D$", where $0 : \mathbf{1} \to D$ is a given point of D. Then for any endofunctor $F : \mathcal{B} \to \mathcal{B}$ with a (tensorial) strength, we have, by the above construction, $\lambda_{D,M} : F(D \pitchfork M) \to D \pitchfork F(M)$. If $F(M)$ is a manifold whenever M is, then this map is a map between manifolds, since manifolds form a full subcategory of $\mathcal{B}$; and $\lambda_{D,M}$ is natural in M (since λ is); it is the *flow natural transformation* $F(T(M)) \to T(F(M))$ for F considered in [12] 39.1 (denoted there ι_M). It originated in a discussion between Kolář and the present author in the early 1980s, see the "Remarks" at p. 349 in loc.cit.

An application of the flow natural transformation is that it gives a "prolongation procedure" for vector fields on M: to a vector field $\xi : M \to T(M)$ on M, one constructs a vector field $\tilde{\xi} : F(M) \to T(F(M))$ on $F(M)$, namely the composite

$$F(M) \xrightarrow{F(\xi)} F(T(M)) \xrightarrow{\lambda_{D,M}} T(F(M)).$$

11 Jet bundles

The kth order jet bundle of a smooth fibre bundle $p : E \to B$ in differential geometry is another smooth fibre bundle $J^k(p) \to B$ (usually just denoted $J^k(E)$). The fibre over $b \in B$ consist of k-jets at b of sections of $E \to B$, see [15], or [9] 2.7 (and Remark 7.3.1); in the latter synthetic context, the notion of k-jet becomes representable, in the sense that there is for every $b \in B$ a subset $\mathfrak{M}_k(b) \subseteq B$ (with $b \in \mathfrak{M}_k(b)$), such that a k-jet at b is a map with domain $\mathfrak{M}_k(b)$, in particular, a k-jet of a section of $p : E \to B$ is a map $s : \mathfrak{M}_k(b) \to E$ with $p \circ s$ equal to the inclusion map $\mathfrak{M}_k(b) \to B$. For fixed B, $J^k(E)$ depends in a functorial way on E in the category of smooth fibre bundles over B; synthetically, if $f : E \to E'$ is a map of bundles over B and s is a k-jet of a section of E, $J^k(f)$ takes a section $s : \mathfrak{M}_k(b) \to E$ of E to the map $f \circ s : \mathfrak{M}_k(b) \to E'$. Or, in classical set-up, post-composition by f of a partial section s, representing the given jet, has as k jet at b the desired jet section of E'.

So for each $B \in \mathcal{B}$, we have an endofunctor J^k on the category $\mathcal{B}_B$ (= the category of smooth fibre bundles over B). We shall investigate the functorality properties of J^k as B varies over the category of smooth manifolds.

To simplify the exposition, we shall embed the category of smooth manifolds into a topos model $\mathcal{B}$ of synthetic differential geometry (cf. e.g. [8] or [9]), where the jet construction works not just for smooth fibre bundles $E \to B$ but for any smooth map $E \to B$, (where B is a manifold), so that for each B, J^k is an endofunctor J^k_B on $\mathcal{B}/B$.

The description (from [9] Remark 7.3.1) of J^k is given in terms of the locally Cartesian closed structure of $\mathcal{B}$, as follows: The data of the $\mathfrak{M}_k(b)$, as b ranges over B, resides in "the kth neighbourhood of the diagonal[3] of B",

$$B_{(k)} \underset{d}{\overset{c}{\rightrightarrows}} B;$$

and similarly for A. Here, $B_{(k)} \subseteq B \times B$ consists of pairs (b, b') with $b' \in \mathfrak{M}_k(b)$, and c and d are the restrictions of the two projections $B \times B \to B$. Similarly for $A_{(k)} \subseteq A \times A$ (where we again denote the two projections by c and d). The map $\alpha \times \alpha : A \times A \to B \times B$ restricts to a map $\overline{\alpha} : A_{(k)} \to B_{(k)}$ (equivalently, any map $A \to B$ restricts, for all $a \in A$, to a map $\mathfrak{M}_k(a) \to \mathfrak{M}_k(\alpha(a))$). Pulling back along $d : B_{(k)} \to B$ defines a functor $d^* : \mathcal{B}/B \to \mathcal{B}/B_{(k)}$, and since $\mathcal{B}$ is locally Cartesian closed, this functor has a right adjoint $\Pi_d : \mathcal{B}/B_{(k)} \to \mathcal{B}/B$. In these terms, the endofunctor J^k on $\mathcal{B}/B$ is just the composite $\Pi_d \circ c^*$.

Theorem 11.1. The functors $(J^k_B)^{op}$ are the fibres of an endofunctor $J^k : (\mathcal{B}^2)^* \to (\mathcal{B}^2)^*$ over $\mathcal{B}$.

Proof. Since we already have the functor J^k on the individual $\mathcal{B}/A$, (for $A \in \mathcal{B}$) it is possible to prove this using the construction Theorem 3.1; however, since the categories and functors have so concrete descriptions, it is also informative to give the construction and proofs ad hoc, using set/family - theoretic descriptions, as in the Remark at the end of Section 6. The construction amounts to a process which to a comorphism f over $\alpha : A \to B$ from $X \to A$ to $Y \to B$ associates a comorphism $J^k f$ over α from $J^k X$ to $J^k Y$. Recall that in the set theoretic description (translating (1.8) into elementwise terms), a comorphism f over α amounts to a family of maps $f_a : Y_{\alpha(a)} \to X_a$, for a ranging over A. Similarly, the required $J^k f$ is to consist of a family $(J^k f)_a : (J^k Y)_{\alpha(a)} \to (J^k X)_a$. An element s in $(J^k Y)_{\alpha(a)}$ is a partial section $s : \mathfrak{M}_k(\alpha(a)) \to Y$ of $Y \to B$. The composite

$$\mathfrak{M}_k(a) \xrightarrow{\alpha} \mathfrak{M}_k(\alpha(a)) \xrightarrow{s} Y$$

[3] The use of a "kth neighbourhood of the diagonal", also called "prolongation spaces", for the consideration of jet bundles is crucial in [13]; the setting there is that of manifolds equipped with a structure sheaf of rings (that may contain nilpotent elements), as considered by Grothendieck and Malgrange.

is a map $\mathfrak{M}_k(a) \to Y$ over α, or, equivalently, a map $\mathfrak{M}_k(a) \to \alpha^*(Y)$ over A, thus an element of $(J^k(\alpha^* Y))_a$, to which we may apply the map $J^k f : J^k\alpha^*(Y) \to J^k X$; this means just: post-composing with $f : \alpha^* Y \to X$. Thus, the element in $(J^k X)_a$ that we get is the map $\mathfrak{M}_k(a) \to X$ given elementwise as follows: to input $a' \in \mathfrak{M}_k(a)$, we get as output $f_{a'}(s(\alpha(a'))) \in X_{a'}$. From this later description, the compatibility of the construction of J^k with composition of comorphisms is almost immediate. Q.E.D.

The functors $J^r : (\mathcal{B}^2)^* \to (\mathcal{B}^2)^*$ are not in general Cartesian.

Remark 11.2. Let us note that if the fibres of $Y \to B$ carry some algebraic structure, say that of vector spaces, then so do the fibres of $J^k Y$. This follows, since $J^k = \Pi_d \circ c^*$ is a composite of two right adjoints, so preserves algebraic structure. So $J^k : (\mathcal{B}^2)^* \to (\mathcal{B}^2)^*$ lifts to a functor $\mathcal{V}^* \to \mathcal{V}^*$ over $\mathcal{B}$, where $\mathcal{V} \to \mathcal{B}$ is the category of vector bundles. Similarly for other kinds of algebraic structure, e.g. *pointed* spaces.

Remark 11.3. Let us also remark that the existence of the maps $\alpha^*(J^k Y) \to J^k(\alpha^* Y)$ considered above implies the existence of a fibrational strength of the functor J^k: just take α to be the projection $Q \times B \to B$.

12 Bundle valued 1-forms

The natural setting for the present subsection is the fibration of *pointed* bundles (with morphisms preserving the given points); there is a forgetful functor from the fibration $\mathcal{V} \to \mathcal{B}$ of *vector* bundles to the codomain fibration $\mathcal{B}^2 \to \mathcal{B}$, and this functor factors through the fibration of pointed bundles, but in order not to overload the exposition with too much terminology and notions, the presentation that we shall give is in terms of the fibration $\mathcal{V} \to \mathcal{B}$ of *vector* bundles. If $E \to B$ is such a bundle, a 1-jet at $b \in B$ of a section, i.e. a partial section $s : \mathfrak{M}_1(b) \to E$ is called an *E-valued* (cominatorial) *cotangent* at b if $s(b) = 0_b$. So s and the zero section agree on $b \in \mathfrak{M}_1(b)$, but do not necessarily agree on the whole of $\mathfrak{M}_1(b)$. Clearly, the set of E-valued cotangents form a sub-bundle of $J^1(E) \to B$, called the *bundle of E-valued 1-forms*; let us denote it $\Omega^1(E) \to B$. The functor $E \mapsto \Omega^1(E)$ is a subfunctor of the functor $J^1 : \mathcal{V}^* \to \mathcal{V}^*$ over $\mathcal{B}$.

Let R be a fixed vector space (typically, the ground field). There is a functor $\mathcal{B} \to \mathcal{V}$ over $\mathcal{B}$, assigning to $B \in \mathcal{B}$ the constant vector bundle $B \times R \to B$. This functor is Cartesian, and hence may equally well be viewed as a functor $\mathcal{B} \to \mathcal{V}^*$, since the category of Cartesian arrows in $\mathcal{V}$ and $\mathcal{V}^*$ are the same. Composing with Ω^1,

$$\mathcal{B} \xrightarrow{- \times R} \mathcal{V}^* \xrightarrow{\Omega^1} \mathcal{V}^*,$$

we get a vector star bundle functor, i.e. a functor $\mathcal{B} \to \mathcal{V}^*$; for $\mathcal{B}$ the category of finite dimensional manifolds, it is (isomorphic to) the cotangent bundle functor $T^\dagger : \mathcal{B} \to \mathcal{V}^*$ described Section 8. In algebraic geometry, one sometimes has to *define* the bundle functor $T : \mathcal{B} \to \mathcal{V}$ (tangent bundle) as the composite

$$\mathcal{B} \xrightarrow{\Omega^1} \mathcal{V}^* \xrightarrow{\dagger} \mathcal{V}.$$

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